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AFDELING ZUIVERE WISKUNDE
(DEPARTMENT OF PURE MATHEMATICS)

ZN 104/83 MEI

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PSEUDOCOMPACTNESS FOR G-SPACES

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BIBLIOTHEEK MATHEMATISCH CENTRUM
AMSTERDAM

Printed at the Mathematical Centre, Kruislaan 413, Amsterdam, The Netherlands.

The Mathematical Centre, founded 11 February 1946, is a non-profit institution for the promotion of pure and applied mathematics and computer science. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics Subject Classification: 54H15, 54D35

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Pseudocompactness for G-spaces

by

J. de Vries

ABSTRACT

In this note we prove that if G is a locally compact group and $\langle X,\pi\rangle$ is a Tychonov G-space, then the notions of G-pseudocompactness for $\langle X,\pi\rangle$ and pseudocompactness for X coincide. We also discuss situations where pseudocompactness is implied by the equality $\beta_G X = \beta X$.

KEY WORDS & PHRASES: G-*space*, G-*compactification*, *pseudocompact*, G-*pseudocompact*

In this note we discuss the relationship between the notion of pseudocompactness for G-spaces and two notions of G-pseudocompactness which were introduced independently by S.A. ANTONYAN [1] and the author [6], respectively. It turns out that if G is locally compact, then G-pseudocompactness according to [6] is equivalent with pseudocompactness. The notion of G-pseudocompactness according to [1] is weaker, but in certain special cases it is also equivalent with pseudocompactness. The main result of this note solves several problems of [4] in a rather obvious way: this will be discussed at the end of this note.

For a general theory of G-spaces (= topological transformation groups with acting group G) we refer to [4]. For the convenience of the reader we include here a few definitions from [4] and [6]. The symbol G stands always for a topological Hausdorff group (the Hausdorff property is rather inessential and may without restriction of generality always by assumed as long as we consider actions of G on T_1-spaces: one can always pass to G/G_0 as the acting group, where G_0 is the isotropy subgroup of G).

A G-*space* is a pair $\langle X,\pi\rangle$ where X is a topological space and π (the *action* of G on X) is a continuous mapping from $G \times X$ onto X satisfying the following conditions:

(i) $\pi(e,x) = x$ for all $x \in X$ (e is the unit element of G);

(ii) $\pi(s,\pi(t,x)) = \pi(st,x)$ for all $s,t \in G$ and $x \in X$.

Note, that these axioms imply that for each $t \in G$ the mapping $\pi^t: x \mapsto \pi(t,x): X \to X$ is a homeomorphism. For brevity, we shall write tx for $\pi(t,x)$ $(=\pi^t x)$, Ux for $\{tx: t \in U\}$, etc.. If $\langle X,\pi\rangle$ and $\langle Y,\sigma\rangle$ are G-spaces, then a mapping $\phi: X \to Y$ is called *equivariant* whenever $\phi\circ\pi^t = \sigma^t\circ\phi$ for all $t \in G$, that is, $\phi(tx) = t\phi(x)$ for all $t \in G$ and $x \in X$. Every G-space $\langle X,\pi\rangle$ has an essentially unique unique *maximal G-compactification*

$$\phi_{\langle X,\pi\rangle}: \langle X,\pi\rangle \to \langle \beta_G X,\pi\rangle,$$

that is, an equivariant continuous mapping $\phi_{\langle X,\pi\rangle}$ from $\langle X,\pi\rangle$ to a G-space $\langle \beta_G X,\bar{\pi}\rangle$ where $\beta_G X$ is a compact Hausdorff space, which is characterized by the following property: every equivariant continuous mapping from $\langle X,\pi\rangle$ to a compact Hausdorff G-space factorizes uniquely over $\phi_{\langle X,\pi\rangle}$; cf. [4; 4.3.2(vi)]. If G is locally compact, then the mapping $\phi_{\langle X,\pi\rangle}$ is a dense equivariant embedding of X into $\beta_G X$ iff the space X is Tychonov [5]. So henceforth *we shall assume that* G *is locally compact and that every* G*-space* $\langle X,\pi\rangle$ *has* X *a Tychonov space*. In that case, we shall consider X just as a dense invariant subset of $\beta_G X$, and we write

$$UC^*\langle X,\pi\rangle := \{f|_X : f \in C(\beta_G X)\}.$$

The members of the function space $UC^*\langle X,\pi\rangle$ can be characterized as follows [5]: if $g \in C^*(X)$ (:= the space of bounded real valued functions on X) then $g \in UC^*\langle X,\pi\rangle$ iff

$$(*) \qquad \forall\varepsilon>0 \quad \exists U \in V_e : |g(tx)-g(x)| < \varepsilon \text{ for all } (t,x) \in U \times X.$$

(Here V_e denotes the nbd filter of e in G). The set of all $g \in C(X)$ satisfying condition (*) will be denoted by $UC\langle X,\pi\rangle$ and will be called the set of π-*uniformly continuous functions* (so the elements of $UC^*\langle X,\pi\rangle$ are the *bounded* π-uniformly continuous functions).

A natural question to ask is, under which additional conditions one has $\beta_G X = \beta X$, where βX denotes the ordinary Stone-Čech compactification. (By the equality $\beta_G X = \beta X$ we mean that there exists a homeomorphism h of βX onto $\beta_G X$ such that $h\circ\beta_X = \phi_{\langle X,\pi\rangle}$; here β_X is the canonical inclusion mapping of X into βX.) In general, one has $\beta_G X \neq \beta X$ (see [4;4.4.14 & 4.4.19]), but if, for example, the action of G on X is trivial (that is, $tx = x$ for all $t \in G$ and $x \in X$) or if G is a discrete group [4;7.3.10(iii)], then $\beta_G X = \beta X$. In [1] the following result is announced for compact groups and, unaware of this, I proved it in [6] for arbitrary k-groups:

THEOREM 1. *If* $\langle X,\pi\rangle$ *is a Tychonov* G*-space, and* X *is pseudocompact, then* $\beta_G X = \beta X$. □

The converse is not true: in [1] is a simple example, and here is another one: let X be an arbitrary space and let π be the trivial action of G on X; then $\beta_G X = \beta X$, but X need not be pseudocompact. Actually, this shows that it is improbable to find a simple condition on G or on X which, together with the condition $\beta_G X = \beta X$ will imply that X is pseudocompact:

one needs also a certain non-triviality condition for the action. Since the above counterexample also works for non-trivial actions of *discrete* groups, one might also expect that a certain non-discreteness condition for G would help.

The following result generalizes Theorem 4.10 of [2] (local compactness of G need not be assumed).

THEOREM 2. *Let* $\langle X,\pi\rangle$ *be a* G-*space with* X *a* T_4-*space. If* $\beta_G X = \beta X$, *then for every net* $\{(t_\lambda,x_\lambda)\}_{\lambda\in\Lambda}$ *in* $G \times X$ *such that* $t_\lambda \rightsquigarrow e$ *in* G *one has* $\overline{\{x_\lambda : \lambda \in \Lambda\}} \cap \overline{\{t_\lambda x_\lambda : \lambda \in \Lambda\}} \neq \emptyset$.

PROOF. Suppose that two closed sets as indicated in the statement of the theorem are disjoint. Then they have disjoint closures in βX. By passing to a suitable subnet, we may assume that the net $\{x_\lambda\}_{\lambda\in\Lambda}$ converges to a point z in βX. Since the action of G on X extends to a continuous action of G on $\beta X (=\beta_G X$ by assumption) and the net $\{t_\lambda\}_{\lambda\in\Lambda}$ converges to e in G, it follows that $t_\lambda x_\lambda \rightsquigarrow ez = z$. This contradicts the disjointness of the closures in βX of the two sets indicated above. □

The following corollary of this theorem may be seen as a modification of Proposition 3.4 of [3] (one of the difficulties which prevent a honest generalization of that result to the present context is, that the mappings $\pi_x : t \mapsto tx : G \to X$ are in general not open). Recall, that if α is a cardinal number, then a space is called α-*pseudocompact* whenever every locally finite family of mutually disjoint, non-empty open subsets has cardinality less than α. The *local weight* of G (i.e. the least cardinal number of a local basis of G at e) will be denoted by $\ell w(G)$. Finally, recall that if $\langle X,\pi\rangle$ is a G-space, then the isotropy subgroup of x in G is the subgroup $G_x := \{t \in G : tx = x\}$. If X is a T_1-space, then G_x is always closed in G, because the mapping $\pi_x : t \mapsto tx : G \to X$ is continuous.

COROLLARY 1. *Let* $\langle X,\pi\rangle$ *be a* G-*space with* X *a* T_4-*space such that* $\beta_G X = \beta X$. *Then either the set*

$$X_0 := \{x \in X : G_x \text{ is open in } G\}$$

has a non-empty interior, or X *is* $\ell w(G)$*-pseudocompact.*

PROOF. Suppose the contrary: there exists a dense set of points in X, each having non-open isotropy group, and X is not $\ell w(G)$-pseudocompact. Then there exists a locally finite, disjoint family $\mathcal{W}$ of non-empty open subsets of G with cardinality $\ell w(G)$. Let $\mathcal{B}$ be a local basis at e having cardinality $\ell w(G)$, and let $U \mapsto W_U$ be an injective mapping from $\mathcal{B}$ into $\mathcal{W}$. For every $U \in \mathcal{B}$ there exists a point x_U in W_U with non-open isotropy group. So there exists $t_U \in U$ such that $t_U x_U \in W_U$ and $t_U x_U \neq x_U$. Since the family $\{W_U : U \in \mathcal{B}\}$ is locally finite, the sets $\{x_U : U \in \mathcal{B}\}$ and $\{t_U x_U : U \in \mathcal{B}\}$ are closed in X. Since they are also disjoint, this contradicts the theorem above. □

REMARK. Observe, that in the proof of this corollary local finiteness of the family $\{W_U : U \in \mathcal{B}\}$ is not very essential. Indeed the set $\{x_U : U \in \mathcal{B}\}$ is disjoint from the closure of $\{t_U x_U : U \in \mathcal{B}\}$, because the neighbourhood W_U of x_U contains only the element $t_U x_U$ of the latter set; similarly, the other way round. So it would be sufficient for the proof to guarantee that one of the sets is closed. Thus, if we define

$$s_G(X) := \sup\{\text{card } M : M \subseteq X \sim X_0 \text{ and } M \text{ is discrete and closed in } X\}$$

then a similar proof shows that

$$s_G(X) < \ell w(G).$$

It is not difficult to see, that X_0 is an invariant subset of X (indeed, for $t \in G$ and $x \in X$ we have $G_{tx} = tGt^{-1}$). Moreover, all invariant points belong to X_0. If G is connected, the only open subgroup of G is G itself, so in that case X_0 equals exactly the set of all invariant points in X. If X_0 has empty interior, then we shall say that the G-space $\langle X,\pi\rangle$ *has almost no open isotropy groups.*

COROLLARY 2. *Let* G *be locally compact and let* $\langle X,\pi\rangle$ *be a* G *space with almost no open isotropy groups. If, in addition,* X *is a separable metric space, then the equality* $\beta_G X = \beta X$ *implies that* X *is pseudocompact, hence compact.*

PROOF. We may assume that G acts effectively on X. (Otherwise, pass to the corresponding effective action of G/G_0, where $G_0 := \bigcap \{G_x \,\vdots\, x \in X\}$; observe, that G/G_0 is locally compact, and that for given $x \in X$ the isotropy subgroup in G is open iff the corresponding isotropy subgroup in G/G_0 is open.) Then $\ell w(G) \leq w(X)$ [4;1.1.23], so by Corollary 1, X is pseudo-$w(X)$-compact. In our case, however, $w(X) = \aleph_0$, and pseudo-$\aleph_0$-compactness is the same is ordinary pseudocompactness. □

REMARK. If G is locally compact, non-discreet, and G acts freely on a metric space X (ie. $G_x = \{e\}$ for every $x \in X$) then also $\beta_G X = \beta X$ implies that X is (pseudo)compact. For still another case where $\beta_G X = \beta X$ implies pseudocompactness of X, see Corollary 4 below.

In [1], a Tychonov G-space $\langle X,\pi \rangle$ such that $\beta_G X = \beta X$ was called G-pseudocompact. Unfortunately, in [6] I introduced a different notion of G-pseudocompactness (that it is really different follows from the examples above and theorem 3 below). The notion of G-pseudocompactness according to [6] is as follows:

Let $\langle X,\pi \rangle$ be a Tychonov G-space. A finite (resp. countably infinite) collection $\mathcal{B}$ of mutually disjoint, non-empty open subsets in X is called a G-*dispersion* whenever it satisfies the following condition:

$$(**) \qquad \exists U \in V_e \,\vdots\, \forall B \in \mathcal{B}\ \exists x_B \subseteq B \,\vdots\, Ux_B \subseteq B.$$

The G-space $\langle X,\pi \rangle$ is called G-*pseudocompact* whenever every locally finite G-dispersion in X is finite. It is obvious, that if G is discrete or if the action of G on X is trivial, then G-pseudocompactness of $\langle X,\pi \rangle$ is exactly the same as pseudocompactness of X. Moreover, if X is pseudocompact, then $\langle X,\pi \rangle$ clearly is G-pseudocompact, but the converse was left as an open problem in [6]. In [6;5.8] I conjectured that the converse is false, but I could find no counterexample. The following theorem shows, why I couldn't; the proof is quite simple.

THEOREM 3. *Let* G *be locally compact and let* $\langle X,\pi \rangle$ *be a Tychonov* G-*space. Then* $\langle X,\pi \rangle$ *is* G-*pseudocompact iff* X *is pseudocompact.*

PROOF. "If": obvious (see also the remarks above). "Only if": let $\{W_n\}_{n \in \mathbb{N}}$

be an infinite sequence of non-empty open subsets of X, mutually disjoint. Let U be a compact symmetric neighbourhood of e in G and let $x_n \in W_n$ for every $n \in \mathbb{N}$. Since $\langle X,\pi\rangle$ is assumed to be G-pseudocompact, no sequence $\{W'_n\}_{n\in\mathbb{N}}$ with W'_k an open neighbourhood of Ux_k for every $k \in \mathbb{N}$ can be locally finite (if there would be such a sequence which is locally finite, then there would also be such a sequence which is *disjoint* and locally finite, i.e. a locally finite G-dispersion; for the straightforward proof of this, see [6;2.2(4^0)]. In particular, the sequence $\{UW_n\}_{n\in\mathbb{N}}$ is not locally finite: there exists a point x_0 in X such that every neighbourhood V of x_0 intersects infinitely many of the sets UW_n. Let V be a neighbourhood of Ux_0. Since the action of G on X is continuous as a mapping of $G \times X$ into X and U is compact, there exists a neighbourhood V' of x_0 such that $UV' \subseteq V$. For infinitely many values of $n \in \mathbb{N}$ we have now that $V' \cap UW_n \neq \emptyset$, hence $UV' \cap W_n \neq \emptyset$ (for $U^{-1} = U$), and, consequently, $V \cap W_n \neq \emptyset$. If the sequence $\{W_n\}_{n\in\mathbb{N}}$ were locally finite, then the compact set Ux_0 would have a neighbourhood, intersecting only finitely many of the sets W_n. Thus, the sequence $\{W_n\}_{n\in\mathbb{N}}$ is not locally finite. This shows, that X is pseudocompact. □

Using this theorem, we now reformulate some results from [6]; in doing so, some of the open problems of [6] are solved.

COROLLARY 3. *Let* G *and* $\langle X,\pi\rangle$ *be as in the theorem above. Consider the following properties:*

(i) *Every* $f \in UC^*\langle X,\pi\rangle$ *has a maximum and a minimum on* X;

(ii) X *is pseudocompact*

(iii) *Every* π*-uniformly continuous function on* X *is bounded.*

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) *and* (iii) $\not\Rightarrow$ (ii).

PROOF. For (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (iii) $\not\Rightarrow$ (ii), see [6;2.5]. The implication (ii) $\Rightarrow$ (i) is trivial. □

REMARK. In [6; Remark 5.11] the implication $\langle X,\pi\rangle$ is G-pseudocompact $\Rightarrow$ (i) was left open. A problem which was not considered in [6] is, under which additional conditions one has (iii) $\Rightarrow$ (ii) in the above corollary. Here is a partial solution:

COROLLARY 4. *Let* G *be a locally compact metrizable topological group, and let* $\langle X,\pi\rangle$ *be a normal Hausdorff* G*-space. Assume that there are almost no* *open isotropy subgroups. Then the following conditions are equivalent:*

(i) X *is pseudocompact;*

(ii) *Every π-uniformly continuous function on* X *is bounded and* $\beta_G X = \beta X$.

PROOF. For (i) $\Rightarrow$ (ii), see Corollary 3 together with Theorem 1. For (ii) $\Rightarrow$ (i), suppose f is an unbounded π-continuous function. Without restriction of generality we may suppose that $f \geq 0$. Let $\{x'_n\}_{n\in\mathbb{N}}$ be a sequence in X such that $f(x'_{n+1}) > f(x'_n)+1$ for all n, and let $W_n := \{x \in X : |f(x)-f(x'_n)| < 1/3\}$. Since $\langle X,\pi\rangle$ has almost no open isotropy groups, for every $n \in \mathbb{N}$ there is a point $x_n \in W_n$ such that the isotropy group G_{x_n} is not open in G. Now the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a discrete, closed subset, and (as G is metrizable) we can find a sequence $\{t_n\}_{n\in\mathbb{N}}$ in G such that $t_n x_n \neq x_n$, $t_n x_n \in W_n$ for all n, and $t_n \rightsquigarrow e$ for $n \to \infty$ (cf. the proof of Corollary 1). As in the proof of Corollary 1 (see also the Remark after that proof), this contradicts Theorem 2. □

REMARKS 1. Problem 5.3 of [4] remains open.

2. The general question for necessary and sufficient conditions for the equality $\beta_G X = \beta X$ is still open. The problem whether G-speudocompactness is sufficient (cf. [6;5.10] is solved by Theorems 1 and 3 above: the answer is "yes". In this context, see also Theorem 6 in [1].

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